# *Huygens, Holland, and Hanging Chains, or L'affaire de la chaîne*

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I need to begin with a disclaimer and an apology. The advertisement for this lecture began with the words, "At the age of 17, Dutch mathematician Christiaan Huygens proved a theory of Galileo to be wrong...." This is not exactly correct, as it was more of an aside or an afterthought of Galileo that Huygens proved to be wrong. In his Discorsi of 1638, Galileo was writing about strength of materials and cross-sections of beams, when the curve known as the parabola kept appearing. After correctly explaining how to draw such a curve, he stated, "The other method of drawing the desired curve ... is the following: Drive two nails into a wall at a convenient height and at the same level.... Over these two nails hang a light chain.... This chain will assume the form of a parabola."[2] An example of a parabola is shown in Figure 1. The equation of the simplest parabola is $y = x^2$, and the general equation of such a curve may be given by $y = ax^2 + x + c$. We could also discuss the parabola in more geometric terms, referring to the focus and the directrix, but we won't do that here. The shape of this curve is what is important to us.

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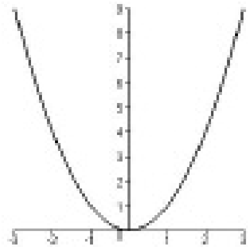

Figure 1. The parabola $y = x^2$

If we follow Galileo's instructions and hang such a chain, either over two nails or with two hooks attached to suction cups, it certainly looks like it could be a parabola. In fact, Galileo was merely stating what was commonly thought about the problem of the hanging chain. It was widely accepted in the early seventeenth century that a hanging chain did take the form of a parabola. The problem is thought to have first appeared about a century earlier, when Leonardo da Vinci sketched a few hanging chains in his notebooks.[3] Other prominent seventeenth-century mathematicians also considered this problem. In 1614, the Flemish mathematician Isaac Beeckman (1588-1637) wrote in his notebook, "Let there hang from a beam a cord…attached at the ends so that it hangs loosely and freely,"[4] while including an accompanying sketch. Beeckman also asked his friend René Descartes (1596-1650) about it. In the writings of Descartes, there is a note, "Sent to me by [Beeckman] whether a cord…affixed by nails…may describe part of a conic section [or a parabola]."[5] There is no indication that Descartes attempted to solve the problem or even that he discussed it further with Beeckman.

The problem of the hanging chain also appeared in the works of the famous Dutch mathematician Simon Stevin (1548-1620). Stevin's *Les Oeuvres Mathématiques* was published posthumously in 1634, with annotations by the French mathematician Albert Girard (1595-1632). It is believed that Christiaan Huygens may have learned of the problem here, since the works of Stevin were recommended to him in 1645 by Stampioen de Jonge,[6] hired by Huygens's father Constantijn to provide mathematical instruction for his two oldest sons. Near the end of Stevin's *Oeuvres*, Girard wrote, "because the other slack or taut ropes are parabolic lines (as

I have in the past proven around the year 1617), as I will prove after this at the end of the following corollary, that which will be strongly appropriate for the ornamentation of these Spartostatique."[7] It appears that there was no proof from the year 1617, however, as Girard continues after Stevin's corollary, "To satisfy my last promise which precedes the last corollary, and not having the spare time however to place here a copy of my entire proof, I will give it another time in public, with my other works, in return for the help of God, when the research of the sciences will be very commendable, which it is not at present.... *Fin de la Spartostatique*."[8] So the work ends without an appearance of a proof from Girard, who was also dead at the time of the publication of Stevin's *Oeuvres*.

Now we meet our main characters, Christiaan Huygens (1629-1695) and Marin Mersenne (1588-1648). Christiaan Huygens was the second son of Constantijn Huygens in an important Dutch family. Constantijn was well-known in Holland as a composer and a poet. He worked in the service of the government, was somewhat wealthy, and had important friends. Father Marin Mersenne was a French monk in the order of the Minims. He was a scientist and a correspondent to many other scientists and intellectuals throughout Europe. In fact, after Mersenne's death, letters from more than seventy such thinkers were found in his Paris cell. One of Mersenne's correspondents was Constantijn Huygens.

The importance of the Huygens family is evident throughout Holland. There is a painting of Constantijn Huygens and his wife (Christiaan's mother) Susanna van Baerle, by J. van Campen, in the Mauritshuis in The Hague. Also in this small yet wonderful museum is the famous portrait of Constantijn Huygens and his five children, by A. Hanneman. This painting appears in nearly every book about Christiaan Huygens. Turning to street names, there is the Eerste Constantijn Huygensstraat in Amsterdam. (Eerste means "first," indicating that the street is named for the first Constantijn Huygens, Christiaan's father, as opposed to the second Constantijn Huygens, Christiaan's brother.) Of course, there is also the Rue Huyghens in the 14th arrondissement of Paris, where I made a pilgrimage during my six-hour visit there last summer. Of course, this street is most likely named for Christiaan himself, as the city of Paris is filled with streets named for scientists, writers, and musicians.

The focus of my work on the history of the problem of the hanging chain is a series of letters between Christiaan Huygens and Marin Mersenne, beginning in 1646 and ending in 1648 with Mersenne's death. As mentioned earlier, Constantijn Huygens was a correspondent of Mersenne, and he was proud to tell Mersenne about his brilliant seventeen-year-old son Christiaan. It was Mersenne who initiated the correspondence with Christiaan in a letter dated October 13, 1646. The letter dealt with a different mathematical topic, about which Mersenne wrote, "As I greatly respect your father, whom I believe is pleased to speak of your propositions of which you say to have a proof, I will say only of the last one, that I do not believe that you have a proof, as I have not seen one...."[9] Two weeks later, on October 28th, Christiaan responded to Mersenne's initial letter with a full explanation of the other problem, finishing his letter with the additional promise, "I will send you in another letter a proof that a hanging cord or chain does not make a parabola, and what should be the pressure on the mathematical cord or one without gravity to make one; I have found such a proof not long ago."[10]

We note here that Huygens was in fact correct in saying that the hanging chain does not make a parabola. We call the curve formed by such a chain a "catenary," a term originally coined by Huygens himself in a 1690 letter to Gottfried Leibniz. It is interesting to note that our word "catenary" is derived from the Latin *catena*, meaning "chain." So we say that the shape of a hanging chain is…a chain!

After receiving another letter in which Mersenne expressed great interest in seeing Christiaan's discussion of the hanging chain, Huygens then sent him the requested proof in November 1646. He began with four assumptions (or "axiomata"), two of which are the following:[11]

> Assumption 1. I suppose therefore first that the whole cord depends only on some gravity, tending toward the center of the earth, to be parallel to one another.
> Assumption 2. Secondly, that two or more weights…pull on the cord…which is held at [two endpoints].

Huygens then begins to set up his argument with the following proposition and the corresponding Figure 2.

> Proposition 5.[12] If there are so many weights that one wants S, R, P, Q hanging from a cord ABCD, I say that MD and BC continued intersect at L on the hanging diameter of the weights P and Q. AB and DC [intersect] at K on the hanging diameter of the weights R and P, and in this way the rest....

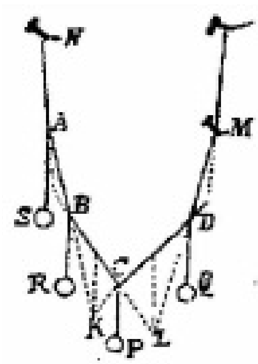


Figure 2.[13] Huygens's sketch to accompany Proposition 5

In Proposition 5, Huygens sets up a weightless chain with individual weights hanging from it. He discusses the form taken by the chain, explaining how the extensions of certain segments in Figure 2 intersect on the "hanging diameter of the weights" between them. (I want to point out that this phrase, "hanging diameter of the weights," is awkward in French, awkward in Latin, and therefore also awkward in English!) What this means is that these extensions intersect at a point on the vertical line that is halfway between the two weights and therefore also bisects the segment on the chain above. In Proposition 6, Huygens removes the hanging weights and instead considers weighted line segments making up the chain. He argues in a similar fashion that the chain hangs exactly the same way as in Proposition 5. These propositions are based on an earlier result known simply as Stevin's Theorem.

At this point, Huygens is nearly ready to state the heart of the proof. Before we examine this next proposition, we turn to a statement made by Descartes in a letter to Princess Elizabeth, "In the solution of a geometrical problem . . . I use no theorems except those which assert that *the sides of similar triangles are proportional*, and that in a right triangle the square of the hypotenuse is equal to the sum of the squares of the sides."[14] Huygens had a solid knowledge of geometry and knew the works of Descartes, so it is not surprising that the upcoming argument put forth by Huygens uses similar triangles as an integral part of the proof. We define two triangles to be similar if their corresponding angles are equal, as for the triangles in Figure 3. We then see in Figure 4 that two parallel lines drawn across an angle will create two similar triangles. This is how Huygens incorporates similar triangles into his Proposition 8.

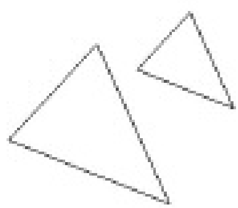

Figure 3. Similar triangles

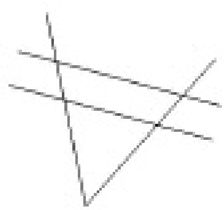

Figure 4. A simple case of similar triangles

> Proposition 8.[15] Let the hanging chain HGABCDK consist of lines of equal length, weight, and shape; I say the points of connection GABCDK cannot coincide on the same parabolic line.

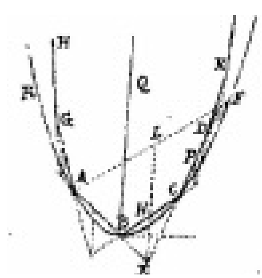

Figure 5.[16] Huygens's sketch to accompany Proposition 8.

Huygens begins his proof[17] of Proposition 8 in the following way:

> From Proposition 6 it is evident in what manner these lines may be bound to hang truly, so that H may be in the middle of BC, P in the middle of CD, etc. And so now the parabola RABCF having been described, passing through three points A, B, C, I say that this is not to pass through the point D and the rest....

Here, Huygens sets up a hanging chain of equal segments and the parabola RABCF in the same diagram, as shown in Figure 5. He uses the information from Propositions 5 and 6 to say that the extensions of the line segments intersect on the hanging diameters of the weights, and he claims that the parabola intersects the chain only at the points A, B, C, and nowhere else. (It is worth noting here that one can always fit a parabola to any three points. He fits the parabola to A, B, C, and he hopes to show that it does not pass through the other points H, G, D, K, on the chain.) If Huygens can go on to prove this claim, he will have shown that the chain and the parabola are indeed different shapes. He continues with the following:

> for ECD may be extended until it may be that $\frac{FC}{CE} = \frac{AB}{BE}$ , and then AF may be drawn, and this therefore will be parallel to BC and similarly will be divided in two by the line EL at L, therefore the point F will be on the same parabola with the points A, B, C, for EL is a diameter of the parabola B, and not the point D.

Huygens is now referring to the definition of diameter put forth by Apollonius in the third century B.C., that a diameter is a line that bisects any set of parallel segments across a parabola. The diameter is always parallel to the axis of the parabola, the line down the "middle" of the parabola. Huygens sets up his first set of similar triangles, $\triangle$FEA and $\triangle$CEB, and he equates the ratios of corresponding sides, which is how one takes advantage of the presence of similar triangles. He claims that the point F is on the given parabola, but that the point D is not. He explains why D is not on the parabola in the final sentence of the proof of Proposition 8:

> For otherwise the line ECDF might have been obliged to cut the parabola in three points, which is absurd, or the point D to coincide with the point F, which is impossible, for FC > AB or DC, since CE > BE.

No matter how hard we try to draw a line that intersects a parabola in three points, it is impossible, so this is not an option. The other option is that D and F are really the same point; then the line would only intersect the parabola in two points. But Huygens uses the equation above, $\frac{FC}{CE} = \frac{AB}{BE}$ to show that this is not possible. From the picture, we see immediately that CE > BE. Using this information in the equation, we get that FC > AB. But AB = DC, since all segments of the chain are equal in length. Therefore, Huygens shows that FC > DC, meaning that F and D are distinct points. With this, Huygens proves that D is not on the parabola. Proposition 8 continues with similar arguments to show that the other points of the chain are not on the parabola either. In this way, Huygens shows that the chain does not hang in the form of a parabola.

Huygens's work on this problem does not end here. His Proposition 9 essentially restates the result of Proposition 8 that "no chain hangs according to a parabolic line."[18] Next, Proposition 10 is a very intriguing statement, in which he says that there is "no notable difference between the line which hangs, and that which may hang if it were composed of equal lines...."[19] Here he is comparing the (smooth) hanging cord and the chain composed of connected straight segments, saying that the smooth cord should also not hang as a parabola. Of course this is a correct statement, but Huygens offers no proof of this in his letter.

In Proposition 11, Huygens explains how to make the chain into a parabola, as he promised in his first letter to Mersenne. He proposes to hang equal weights from a weightless string at equal intervals *along the horizontal* (as opposed to equal intervals along the string itself). In this case, the string does in fact take the form of a parabola. Clearly excited by this, Huygens goes on to propose another way to make a parabola: "Hence it is clear that if on the string…might be placed little beams or parallelepipeds (that is, rectangles) of equal weight, size, and shape, the points…press on the string, each one to be on the same parabola...."[20] Of course, he does not prove this statement, but the sketch he provides looks reasonable enough. Twenty-two years later, however, in the margin of another version of this same argument, Huygens writes, "*non sequitur neque est verum*"[21] —"it does not follow, nor is it true." Here in 1668, Huygens realizes his error in this final statement from his youthful correspondence. In fact, if one stacks rectangles inside a hanging cord, an arc of a circle will result. In the case of the weights hanging from the string (as in Proposition 11), the cord is pulled downward, whereas here the rectangles push the cord outward in a normal direction (certainly not downward!), causing the two situations to produce different shapes.

After Huygens's letters to Mersenne in 1646, the problem of the hanging chain was not studied for many years. The next known attempt to understand the problem was by the Jesuit Father Ignace Gaston Pardies (1636-1673), who considered the problem in *La Statique, ou la Science des Forces Mouvantes*,[22] published in 1673. In this work, Pardies also proves that the hanging chain does not take the shape of a parabola. In contrast to the awkward geometrical arguments of Huygens, the proof by Pardies is much more elegant.

You should have noticed that although Huygens and Pardies were able to show that the hanging chain is not a parabola, they were not able to say what shape the hanging chain actually is. More years passed before Jacob Bernoulli posed the problem in *Acta Eruditorum*[23] in May 1690, challenging readers to come up with the solution of the actual shape of the hanging chain. The *Acta* was a very interesting journal, created and edited by Leibniz, containing articles on mathematics, biology, religion, and philosophy, among other topics. Many of the interesting mathematical problems of the time appeared in the *Acta*. Just over one year later, in the June 1691 *Acta*, four solutions to the problem of the hanging chain appeared—those of Leibniz, Johann Bernoulli, Jacob Bernoulli, and Christiaan Huygens! These papers were not the focus of my recent study, although I hope to read them in the future. One of the things I find most interesting about Christiaan Huygens and the hanging chain is that he worked on this problem at the beginning of his (mathematical) life in 1646 and again near the end of his life, separated by forty-five years and using two completely different approaches.

For those of you who are mathematically inclined, we know today that the shape of the hanging chain, the "catenary," is a hyperbolic cosine, a combination of exponential functions. Although this is well-known among mathematicians now, it took the 17-year-old Christiaan Huygens to first show that the curve was not a parabola, setting the stage for some of the biggest names in late seventeenth-century mathematics to determine the true shape of the curve.

NOTES

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All translations of Huygens, Girard, and Beeckman (from Latin, French, and Dutch) are those of the author, unless otherwise specified.

[2] Galileo Galilei, *Dialogues Concerning Two New Sciences*, trans. by Henry Crew and Alfonso deSalvio, ed. by Stephen Hawking (Philadelphia: Running Press, 2002), 114.

[3] Clifford Truesdell, *The Rational Mechanics of Flexible or Elastic Bodies, 1638-1788*, as Introduction to *Leonhardi Euleri Opera Omnia*, Series II, Volume 11, Part 2 (Zürich: Füssli, 1960), 21. Truesdell's work presents a nice overview of the history of the problem.

[4] Isaac Beeckman, *Journal tenu par Isaac Beeckman de 1604 à 1634*, Tome I (The Hague: Nijhoff, 1939): 43.

[5] René Descartes, in Appendix II of Beeckman, *Journal*, T. I, 362.

[6] Christiaan Huygens, *Oeuvres Complètes*, Tome XI (The Hague: Nijhoff, 1908), 37.

[7] Simon Stevin, *Les Oeuvres Mathématiques* (Leiden: Bonaventure and Elsevier, 1634), 508.

[8] Ibid.

[9] Huygens, *Oeuvres Complètes*, T. I (The Hague: Nijhoff, 1888), 558.

[10] Ibid., 28.

[11] Ibid., 34.

[12] Ibid., 35.

[13] Ibid.

[14] René Descartes, *The Geometry of René Descartes*, trans. by David E. Smith and Martha L. Latham (New York: Dover, 1954), p. 10; originally in René Descartes, *Oeuvres de Descartes*, vol. IX, ed. by Victor Cousin, (Paris: Levrault, 1825), 144.

[15] Huygens, *Oeuvres*, T. I, 36.

[16] Ibid.

[17] Ibid., 36-37.

[18] Ibid., 37.

[19] Ibid., 38.

[20] Ibid., 39.

[21] Ibid., T. XI, 43.

[22] Ignace Gaston Pardies, *Oeuvres du R.P. Ignace-Gaston Pardies* (Lyon: Bruyset, 1725), 280-281.

[23] Jacob Bernoulli, "Analysis problematis antehac propositi, de inventione linea descensus a corpore gravi percurrendae uniformiter, sic ut temporibus aequalibus aequales altitudines emetiatur: & alterius cujusdam Problematis Propositio," *Acta Eruditorum* (May 1690): 217-219.